

Note on uncertainty relations in doubly special relativity and rainbow gravity

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Abstract

We present commutation relations depending on the rainbow functions which are slightly different from the well-known results. However, the advantage of these new commutation relations are compatible with the calculation of the Hawking temperature in the rainbow Schwarzschild black hole.

PACS numbers:04.50.Kd, 04.62.+v

Keywords: Black Hole, Thermodynamics

Typeset Using L^AT_EX

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One of the most interesting issues in quantum theory of gravity is that there appears a minimal length from various points of view such as string theory [1, 2], loop quantum gravity [3], and Gedanken experiments [4, 5, 6]. The minimal length is related to the modification of quantum-commutation relations [7, 8, 9, 10], so that the Heisenberg uncertainty principle is promoted to the generalized uncertainty principle (for a recent extensive review, see Ref. [11]). Meanwhile, it has also been claimed that a modified dispersion relation could come from the deformation of classical relativity so called the doubly special relativity [12, 13, 14, 15] which is the extended version for the special theory of relativity in the sense that both the Plank-scale energy and the speed of light are required to be invariant under any inertial frames throughout the nonlinear realization of the Lorentz transformations on the momentum space. A simple realization of the modified dispersion relation for the doubly special relativity was exhibited in the flat spacetime [16, 17] and the curved spacetime [18].

Now, it is worth noting that the uncertainty principle from quantum mechanics is indeed independent of the dispersion relation from the classical relativity. So, it is not until a commutation relation or uncertainty relation are imposed that one can relate the energy scale defined in the momentum space with a length scale defined in the spacetime coordinates. In this respect, quantum commutation relations for a specific nonlinear realization could be obtained by using the modified generators satisfying the conventional Lorentz algebra in Ref. [17]. Then, for more general nonlinear realizations, the commutation relations to implement the doubly special relativity at the level of quantum mechanics were obtained by introducing space time coordinates as the generators of translations in the linearly transforming energy-momentum variables [19].

On the other hand, one can obtain the black hole temperature by using the uncertainty principle since the emitted particle energy from the black hole is related to the uncertainty relation. For the usual Schwarzschild black hole, the position uncertainty of the emitted particles is assumed to be the radius of the event horizon of $\Delta x \sim r_H$, so that the momentum uncertainty in the spherically symmetric geometry can be obtained

as $p \sim \Delta p \sim 1/r_H$ in virtue of the usual Heisenberg uncertainty relation. Next, using the standard dispersion relation as $E = p$ for the massless case, the Hawking temperature which is proportional to the energy of the emitted particles can be estimated with a calibration factor as $T_H = 1/(4\pi r_H)$. This heuristic derivation was illuminated in the calculation of the temperature of the black hole subject to the generalized uncertainty principle [20]. The key ingredient is that the Hawking temperature is connected with the uncertainty relation for a given dispersion relation, so that one can get various expressions for the temperature depending on commutator relations, which have been used in the thermodynamic analysis of rainbow black holes [21, 22].

In this work, we would like to find quantum-mechanical commutation relations depending on the rainbow functions and identify the relevant uncertainty relation to the calculation of the Hawking temperature. It will be shown that the uncertainty relation between the position and the momentum can be independent of the rainbow functions, so that it gives the same Hawking temperature as that of calculation of the surface gravity in the rainbow Schwarzschild black hole.

Considering a transformation implemented by $U(p_0)$ acting on p_μ as $U(p_0) \cdot p_\mu = (p_0 f(p_0), p_i g(p_0))$, the modified dispersion relation can be written as [16, 17]

$$\eta^{\mu\nu}(U(p_0) \cdot p_\mu)(U(p_0) \cdot p_\nu) = -p_0^2 f^2(p_0) + p_i p_i g^2(p_0) = -m_0^2, \quad (1)$$

where the line element is given as

$$ds^2 = -\frac{1}{f^2} dt^2 + \frac{1}{g^2} dx^i dx^i \quad (2)$$

in the energy-independent coordinates, and the rainbow functions f and g depend on the energy of test particles. Note that $U(p_0) \cdot p_\mu$ can be replaced by \tilde{p}_μ such that the line element can be written in the form of the energy-dependent fashion. In particular, if we assume that the commutation relations are defined as

$$[x^\mu, \tilde{p}_\nu] = \delta_\nu^\mu, \quad (3)$$

then one can express $x^\mu = \partial_{\tilde{p}_\mu}$. As a matter of fact, these are equivalent to the expression in Ref. [19] where the spacetime coordinates are the generators for translations in the

auxiliary linearly transforming energy-momentum variables \tilde{p}_ν . So the energy-dependent coordinates are related to the energy-independent ones as follows

$$\tilde{x}^0 = f^{-1}x^0, \quad \tilde{x}^i = g^{-1}x^i \quad (4)$$

from the flat metric (2), while $\tilde{p}_0 = p_0 f$ and $\tilde{p}_i = p_i g$ are read off from the dispersion relation (1). Now, the commutation relations can be written by the energy-independent variables by using the chain rules,

$$x^0 = (\partial_{\tilde{p}_0} p_0) \partial_{p_0} + (\partial_{\tilde{p}_0} p_i) \partial_{p_i} = \frac{1}{(f + p_0 f')g} [(g + p_0 g') \partial_{p_0} - g' p_\alpha \partial_{p_\alpha}], \quad (5)$$

$$x^i = (\partial_{\tilde{p}_i} p_0) \partial_{p_0} + (\partial_{\tilde{p}_i} p_j) \partial_{p_j} = \frac{1}{g} \partial_{p_i}, \quad (6)$$

where the prime denotes the derivative with respect to p_0 . Using Eqs. (5) and (6), the commutation relations between the original variables are obtained as

$$[x^0, p_0] = \frac{1}{f + p_0 f'}, \quad (7)$$

$$[x^0, p_i] = -\frac{p_i g'}{(f + p_0 f')g}, \quad (8)$$

$$[x^i, p_j] = \delta_j^i \frac{1}{g}, \quad (9)$$

where they are coincident with the results in Refs. [17, 19]. From Eq. (9), the uncertainty relation between the position and the canonical momentum becomes

$$\Delta x \Delta p \geq \frac{1}{2g}. \quad (10)$$

Let us now consider the metric of the rainbow Schwarzschild black hole in order to apply the above uncertainty relation to the calculation of Hawking temperature. The metric is given as [18],

$$ds^2 = -\left(1 - \frac{2GM}{r^2}\right) \frac{dt^2}{f^2} + \left(1 - \frac{2GM}{r^2}\right)^{-1} \frac{dr^2}{g^2} + \frac{r^2}{g^2} d\Omega^2, \quad (11)$$

where the event horizon is $r_H = 2GM$ and the metric is reduced to the flat rainbow metric (2) for $M = 0$. The Hawking temperature in Gravity's Rainbow measured by the asymptotic observer at infinity can be derived from the heuristic method [20]. From

the dispersion relation (1), one can express the energy for the massless particles as $E = (g/f)p = (g/f)\Delta p$, and it becomes $E = (g/f)(2g\Delta x)^{-1} = (2f\Delta x)^{-1}$ by employing the uncertainty principle (10). Thus the rainbow Hawking temperature is written as

$$T_H = \frac{1}{8\pi GMf} \quad (12)$$

with the calibration factor 2π , where $\Delta x \sim 2GM$. By the way, the Hawking temperature can also be derived from the calculation of the surface gravity of the metric (11) [21, 22], which is given as

$$T_H = \frac{1}{8\pi GM} \frac{g}{f}. \quad (13)$$

Note that the temperature (12) is different from the temperature (13) in that the former case is independent of the rainbow function g .

The above calculation will be repeated by assuming a different uncertainty relation from Eq. (10). For this purpose, let us assume commutation relations defined as

$$[\tilde{x}^\mu, \tilde{p}_\nu] = \delta_\nu^\mu. \quad (14)$$

From the inverse representation of Eq. (4), the coordinates are expressed by

$$x^0 = f\tilde{x}^0 = \frac{1}{1 + p_0(f'/f)} [(1 + p_0(g'/g))\partial_{p_0} - (g'/g)p_\alpha\partial_{p_\alpha}], \quad (15)$$

$$x^i = g\tilde{x}^i = \partial_{p_i}, \quad (16)$$

where the energy-dependent coordinates are represented as $\tilde{x}^\mu = \partial_{\tilde{p}_\mu}$ from Eq. (14). So the nontrivial commutation relations between the spacetime coordinates are calculated as

$$[x^0, x^i] = \frac{(g'/g)}{1 + p_0(f'/f)} x^i, \quad (17)$$

and the other nonvanishing commutation relations become

$$[x^0, p_0] = \frac{1}{1 + p_0(f'/f)}, \quad (18)$$

$$[x^0, p_i] = -\frac{(g'/g)}{1 + p_0(f'/f)} p_i, \quad (19)$$

$$[x^i, p_j] = \delta_j^i. \quad (20)$$

Note that the Heisenberg uncertainty relation between the coordinate and the momentum is still valid even in the rainbow regime if we require the standard commutation relations in the energy dependent frame. So the Hawking temperature of the rainbow Schwarzschild black hole is now obtained by using the modified dispersion relation (1) and the standard uncertainty relation from Eq. (20) as

$$T_H = \frac{1}{8\pi GM} \frac{g}{f}, \quad (21)$$

where we used the above mentioned heuristic method by identifying $\Delta x \sim 2GM$, $E = (g/f)\Delta p$. Note that this is exactly the same as the temperature calculated from the surface gravity (13) in Refs. [21, 22].

In conclusion, we obtained the commutation relations (17)–(20) with the arbitrary rainbow functions. Exceptionally, Eq. (20) is independent of the rainbow function, so that the uncertainty relation between the coordinate and the momentum respects the usual Heisenberg uncertainty relation which provides the compatible calculation of the temperature with the Hawking temperature from the surface gravity.

Acknowledgments

This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIP) (2014R1A2A1A11049571).

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